On Nil Clean Group Rings over Metacyclic Groups

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Introduction

Definition 1.1

A ring R is said to be nil clean if every element is the sum of an idempotent and a nilpotent.

Nil clean rings are an important variant of (well studied) clean rings and were originally introduced in Diesl's papers [5,6]. More information about nil clean rings can be found in [2, 3,5,6,7,8,9,10,12, 13, 14].

We focus on the development on nil clean group rings.

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Definition 1.2

We recall that a group ring RG, where R is a ring and G is a group, is defined as $RG := \{\sum_{g\in G} a_g g \mid a_g \neq 0 \text{ for only finitely many } a_g \in R\}$. RG is a ring with addition defined by

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} (a_g + b_g) g,$$

and the multiplication

$$(\sum_{g\in G} a_g g)(\sum_{h\in G} b_g h) = \sum_{g\in G, h\in G} (a_g b_h) g \cdot h.$$

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Known results about nil clean group rings:

Lemma 1.3 [10, Cor 2.8](McGovern et al. 2015)

Let R be a ring and G be an abelian group. Then RG is nil-clean iff R is a nil-clean ring and G is a 2-group.

Lemma 1.4 [12, Prop 2.9](Sahinkaya, Tang, Zhou, 2017)

Group ring RS_3 is a nil-clean ring, iff both R and the matrix ring $\mathbb{M}_2(R)$ are nil-clean rings, where S_3 is the symmetric group of degree 3.

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In [4](Cui, Li, Wang, 2021), completely determined the nil cleanness of the group rings RD_{2n} and RQ_{2n}.

Proposition 1.5

RD_{2n} (RQ_{2n}) is nil clean iff R is nil clean and $n = 2^k 3^l$ with $l \leq 1$.

It is natural to ask the following question as D_{2n} and Q_{2n} are special type of metacyclic groups.

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When is a group ring RG over a metacyclic group nil clean?

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Lemma 2.1 [6, Proposition 3.13]

A finite direct product $\prod_{i=1}^{n} R_i$ is nil clean if and only if each component R_i is nil clean.

Lemma 2.2 [6, Proposition 3.15]

If I is a nil ideal of a ring R, then R is nil clean if and only if R/I is nil clean

Lemma 2.3 [6, Proposition 3.20]

Let R be a commutative ring. Then R is nil clean if and only if R/J(R) is Boolean and $J(R)$ is nil

Preliminaries

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Using these properties, we obtain the following theorem, which allows us to reduce the nil cleanness of RG to that of $\mathbb{Z}_2 G$

Theorem 2.4

Let R be a commutative ring and G be a group. Then RG is nil clean if and only if R is nil clean and $\mathbb{Z}_2 G$ is nil clean.

Criterion for an arbitrary group ring RG to be nil clean

Necessary and sufficient conditions for an arbitrary group ring RG to be nil clean by using the information of the hypercenter of G

First Main Result- Necessary and Sufficient Condition.

Theorem 3.1

Let R be a ring and G be a group. The following are equivalent:
(1) RG is nil clean.
(2) $R(G/H(G))$ is nil clean and the hypercenter $H(G)$ is a 2-group.
(3) $R(G/\mathcal{Z}_n(G))$ is nil clean and for some $n \geq 1$, the n-center $\mathcal{Z}_n(G)$ is a 2-group.
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Nil clean group rings over metacyclic groups

We now turn back to metacyclic group rings.

Definition 4.1

Recall a group G is said to be metacyclic if it has a cyclic normal subgroup L such that G/L is also cyclic.

Thus $G = \langle a, b \mid a^n = b^m = 1, b^{-1}ab = a^r \rangle$.

Theorems 2.4 and 3.1 allow us to always assume $R = \mathbb{Z}_2$ and G is centerless.

Proposition 4.2

If RG is nil clean, then :
(1) $m = 2^k$ with $k \geq 1$,
(2) If $\mathcal{Z}(G) = 1$, then n is odd.

We reduce to the case where

$$G = \langle a, b \mid a^n = b^m = 1, b^{-1}ab = a^r \rangle$$

with $m = 2^k$, n odd, and $\mathcal{Z}(G) = 1$.

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Second Main Result (Necessary Conditions).

Proposition 4.3

If $\mathbb{Z}_2 G$ is nil clean with $\mathcal{Z}(G) = 1$, then n is square-free when $m = 2, 4, 8, 16$. Moreover, if n is a prime power, Then

(1) If $m = 2$, then $n = 3$

(2) If $m = 4$, then $n = 5$

(3) If $m = 8$, then $n = 17$

(4) If $m = 16$, then $n = 17$ or 257.

Some significant examples

Next, we determine whether the corresponding group rings RG are nil clean. The case of $m = 2$, was completely settled in [4] and [12].

Next we consider the cases for $m = 2^k \geq 4$.

We first investigate the simplified case:

$$G = \langle a, b \mid a^p = b^m = 1, b^{-1}ab = a^r \rangle$$

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If $m = 4$, then by Prop 4.3, $p = 5$. Thus,

$$G_{20} = \langle a, b \mid a^5 = b^4 = 1, b^{-1}ab = a^2 \rangle.$$

Example 5.1

$\mathbb{Z}_2 G_{20}$ is nil clean.

Outline of Proof .

Step 1.

$$\mathbb{Z}_2 G_{20} = R \simeq Re_1 \oplus Re_2 \simeq \mathbb{Z}_2 C_4 \oplus Re_2$$

where $e_1 := \hat{a} = \sum_{i=1}^{5} a^i$, $e_2 := 1 + \hat{a}$ are primitive central idempotents.

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Step 2. Since $Re_2/J(Re_2)$ is artinian, it is semisimple. By Wedderburn-Artin theorem,

$$Re_2/J(Re_2) \simeq \bigoplus_{i=1}^{l} M_{n_i}(F_i)$$

with $n_1 \geq n_2 \cdots \geq n_l$ and all F_i finite fields.

Step 3. Using the fact there exists an element of nilpotency 4 in $Re_2/J(Re_2)$ and $\dim_{\mathbb{Z}_2}(Re_2) = 16$, we can show that $Re_2 \simeq M_4(\mathbb{Z}_2)$, which is nil clean, so is our group ring.

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With similar reasoning, and by finding an element that has a nilpotency of 8, we have the following example for $m = 8$.

Example 5.2

Let R be a commutative nil clean ring and $G = \langle a, b \mid a^{17} = b^8 = 1, b^{-1}ab = a^2 \rangle$. Then RG is nil clean.

- Ring decomposition,

$$\mathbb{Z}_2 G \simeq \mathbb{Z}_2 C_8 \bigoplus Re_1 \bigoplus Re_2$$

- $Re_1 \simeq Re_2 \simeq \mathbb{M}_8(\mathbb{Z}_2)$

Some significant examples

For $m = 16$. We skipped the case when $p = 257$. Consider only the group ring when $p = 17$.

Proposition 5.3

Let R be a commutative nil clean ring and $G = \langle a, b \mid a^{17} = b^{16} = 1, b^{-1}ab = a^{3} \rangle$. Then RG is nil clean.

- $\mathbb{Z}_2 G \simeq \mathbb{Z}_2 C_{16} \bigoplus Re_1$.
- $Re_1 \simeq \mathbb{M}_{16}(\mathbb{Z}_2)$.

Some significant examples

We end the talk with an example of when n is a composite odd integer.

Proposition 5.4

Let R be a commutative nil clean ring and $G = G_{60} = \langle a, b \mid a^{15} = b^4 = 1, b^{-1}ab = a^2 \rangle$. Then RG is nil clean.

- $\mathbb{Z}_2 G_{60} \simeq \mathbb{Z}_2 C_4 \bigoplus Re_1 \bigoplus Re_2 \bigoplus Re_3 \bigoplus Re_4$.
- Re_1 is the direct summand of a nil clean ring.
- Re_2 is nil clean since it is a direct summand of $\mathbb{Z}_2 G_{20}$.
- $Re_3 \simeq Re_4 \simeq \mathbb{M}_4(\mathbb{Z}_2)$.

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Future Research Problems

Research Problem 5.5

Verify whether the corresponding group ring (when m = 16 and p = 257) is nil clean.

Conjecture 5.6

If $\mathbb{Z}_2 G$ is nil clean and $\mathcal{Z}(G) = 1$, then $n = |a|$ is a square-free odd number.

Research Problem 5.7

For what primes p is a group ring $\mathbb{Z}_2 G$ over a metacyclic group $G = \langle a, b \mid a^p = b^m = 1, a^b = a^r \rangle$ nil clean, where $m = 2^k, k \geq 5$.

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Thank You !

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